Transient Analysis of Lossy Parabolic Transmission Lines with Nonlinear Loads

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***Abstract*—The exact analytical expressions of the time-domain step response matrix parameters for the lossy parabolic transmission line are developed, therefore extending the range of problems where Allen's method can be applied for the transient analysis of networks consisting of interconnections of linear distributed elements, lumped linear and/or nonlinear elements, and arbitrary sources. For completeness, similar expressions are derived for the lossless parabolic line. In order to demonstrate the versatility of the techniques presented in this paper, we study the transient response of a lossy parabolic line subjected to the following sets of boundary conditions: 1) a unit step input and a linear load, and 2) a trapezoidal pulse generator and a nonlinear load.**

I. Introduction

Despite its applications in electronic packaging design and other areas, the transient behavior of nonuniform two-conductor transmission lines has been addressed only by a limited number of papers [1]–[10]. The reader is referred to a recent paper by the authors [11] for a discussion of [1]–[9]. We would also like to mention the work of Shutt-Aine [10], who studied tapered microstrip lines using a scattering parameter formulation in the time domain. With this approach, however, the nonuniform lines must be discretized along their length, as in [1]–[3] and [7], which is unattractive in terms of computer memory requirement and CPU time.

In this paper, transients on lossy parabolic lines are studied following Silverberg and Wing's approach [12] (revised by Allen [13]), which was originally applied to lossless and lossy but distorsionless uniform lines in [12], [13], and extended to lossless exponential transmission lines by Bouchard *et al.* [11]. This numerical method can be used for the transient analysis of networks consisting of interconnections of linear distributed elements, lumped linear and/or nonlinear elements and arbitrary independent or dependent sources. The overall network is solved in the time domain using convolution techniques. According to [13], each linear subnetwork is characterized in the time domain by a step response matrix $\overleftrightarrow{a}(t)$.

The inverse Laplace transform is used to obtain the exact analytical expressions of the elements of the $\overleftrightarrow{a}(t)$ matrix for the lossy parabolic line, thus widening the range of application of Allen's method [13]. For completeness, similar expressions are derived for the lossless parabolic line. Moreover, complete transient responses of a lossy parabolic line with linear and nonlinear loads are simulated for illustration.

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II. Time-Domain Step Response Matrix of a Lossy Parabolic Line

Ghausi and Kelly [14] obtained the Y parameters (short-circuit admittances) in closed form for a class of nonuniform distributed networks in which the per-unit-length series impedance $Z(s, z)$ and the per-unit-length shunt admittance $Y(s, z)$ can be separated into functions of the Laplace transform variable s and the distance z along the line; moreover, the product $Z(s, z)Y(s, z)$ is independent of z. In other words, these authors restricted their study to networks such that

$$Z(s, z) = \frac{Z(s)}{p(z)} \quad \text{and} \quad Y(s, z) = Y(s)p(z) \tag{1}$$

where $Z(s) = R_0 + sL_0$ and $Y(s) = G_0 + sC_0$ for a lossy line. The function $p(z)$ describes the variations of the four distributed parameters along the line: R_0, L_0, G_0, and C_0 represent their values at $z = 0$.

We consider a parabolic line for which

$$p(z) = \left(1 + \frac{z}{\kappa}\right)^2 \qquad (z + \kappa) \neq 0 \tag{2}$$

where κ is in meters. The "characteristic impedance" of this parabolic line can be written, using (1) and (2), as

$$Z_0(s, z) = \sqrt{\frac{Z(s, z)}{Y(s, z)}} = \frac{\sqrt{\frac{R_0+sL_0}{G_0+sC_0}}}{\left(1 + \frac{z}{\kappa}\right)^2}. \tag{3}$$

Thus, the "characteristic impedance" is complex, with a distinct resistive and reactive part at each point of the line. Following Chang [9], we shall specify the impedance variations of the parabolic line using the high-frequency value of $Z_0(s, z)$ looking at the near end and at the far end.

Substitution of the above expressions into [14, (4)–(16)] yields the short-circuit admittance parameters of the parabolic line in the Laplace transform domain:

$$Y_{11}(s) = \frac{1}{(R_0 + sL_0)}\left[\frac{\Lambda(s)\coth\Lambda(s)}{l} + \frac{1}{\kappa}\right] \tag{4a}$$

$$Y_{12}(s) = Y_{21}(s) = -\frac{\left(1 + \frac{l}{\kappa}\right)}{(R_0 + sL_0)l}\Lambda(s)\operatorname{csch}\Lambda(s) \tag{4b}$$

$$Y_{22}(s) = \frac{\left(1 + \frac{l}{\kappa}\right)^2}{(R_0 + sL_0)}\left[\frac{\Lambda(s)\coth\Lambda(s)}{l} - \frac{1}{(l + \kappa)}\right] \tag{4c}$$

where

$$\Lambda(s) = l\sqrt{(R_0 + sL_0)(G_0 + sC_0)} \tag{5}$$

and l is the length of the line in meters.

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The goal here is to obtain the step response matrix $\overleftrightarrow{a}(t)$ in the time domain, which is related to the short-circuit admittance matrix in the following manner [13]:

$$\overleftrightarrow{a}(t) = \mathcal{L}^{-1}\{\overleftrightarrow{A}(s)\} = \mathcal{L}^{-1}\{(1/s)\overleftrightarrow{Y}(s)\}. \tag{6}$$

It is thus necessary to evaluate the inverse Laplace transform of

$$A_{11}(s) = \frac{Y_0(0)}{s}\sqrt{\frac{(s+\beta)}{(s+a)}}\coth(\tau\sqrt{(s+a)(s+\beta)}) + \frac{1}{sL_0\kappa(s+a)} \tag{7a}$$

$$A_{12}(s) = A_{21}(s) = -\frac{Y_0(0)}{s}\left(1+\frac{l}{\kappa}\right)\sqrt{\frac{(s+\beta)}{(s+a)}} \cdot \operatorname{csch}(\tau\sqrt{(s+a)(s+\beta)}) \tag{7b}$$

$$A_{22}(s) = \frac{Y_0(0)}{s}\left(1+\frac{l}{\kappa}\right)^2\sqrt{\frac{(s+\beta)}{(s+a)}} \cdot \coth(\tau\sqrt{(s+a)(s+\beta)}) - \frac{(l+\kappa)}{sL_0\kappa^2(s+a)} \tag{7c}$$

where

$$Y_0(0) = \sqrt{\frac{C_0}{L_0}} \quad (\mathrm{S}) \tag{8}$$

$$a = \frac{R_0}{L_0} \quad (\mathrm{s}^{-1}) \tag{9}$$

$$\beta = \frac{G_0}{C_0} \quad (\mathrm{s}^{-1}) \tag{10}$$

$$\tau = l\sqrt{L_0C_0} \quad (\mathrm{s}). \tag{11}$$

Even though the line is lossy, the coefficient $(C_0/L_0)^{1/2}$ still appears as it did in the case of the lossless exponential line (see [11, (19), (20), (23)]). This time, $Y_0(0)$ can be interpreted as the high-frequency value of the "characteristic admittance" at $z = 0$ [see (3)].

With the exception of the second term of (7a) and (7c), the inverse transform of the elements $A_{ij}(s)$ in (7) cannot be found directly in two of the most complete tables of Laplace transforms [15], [16]. Accordingly, it seems appropriate to show how the inverse Laplace transforms of these expressions were obtained.

The evaluation of $a_{11}(t) = \mathcal{L}^{-1}\{A_{11}(s)\}$ is considered first. Since [17]

$$\coth x = 2\sum_{n=0}^{\infty}\epsilon_n e^{-2nx} \qquad \mathrm{Re}\,(x) > 0 \tag{12}$$

where

$$\epsilon_n = \begin{cases} 1/2, & n = 0 \\ 1, & n > 0 \end{cases} \tag{13}$$

the first term of (7a) can be expanded into a series:

$$A_{11}(s) = \frac{2Y_0(0)}{s}\sqrt{\frac{(s+\beta)}{(s+a)}}\sum_{n=0}^{\infty}\epsilon_n e^{-2n\tau\sqrt{(s+a)(s+\beta)}} + \frac{1}{sL_0\kappa(s+a)}. \tag{14}$$

Taking the inverse transform of (14) term by term, using [15, p. 253, #3.2–64] and also [15, p. 181, #1–2] and reworking constants (9) and (10), yields

$$a_{11}(t) = 2Y_0(0)\sum_{n=0}^{\infty}\epsilon_n\left[e^{-\rho t}I_0[\sigma\sqrt{t^2-(2n\tau)^2}] + \beta\int_{2n\tau}^{t} e^{-\rho x}I_0[\sigma\sqrt{x^2-(2n\tau)^2}]\,dx\right]u(t-2n\tau) + \frac{(1-e^{-at})}{R_0\kappa}u(t) \tag{15}$$

with

$$\rho = \frac{1}{2}(a+\beta) = \frac{1}{2}\left(\frac{R_0}{L_0}+\frac{G_0}{C_0}\right) \tag{16}$$

and

$$\sigma = \frac{1}{2}(a-\beta) = \frac{1}{2}\left(\frac{R_0}{L_0}-\frac{G_0}{C_0}\right). \tag{17}$$

$I_0(y)$ is the modified Bessel function of the first kind of index zero [16].

The evaluation of $a_{22}(t)$ follows a line similar to that used for $a_{11}(t)$, which gives

$$a_{22}(t) = 2Y_0(0)\left(1+\frac{l}{\kappa}\right)^2\sum_{n=0}^{\infty}\epsilon_n\left[e^{-\rho t}I_0[\sigma\sqrt{t^2-(2n\tau)^2}] + \beta\int_{2n\tau}^{t} e^{-\rho x}I_0[\sigma\sqrt{x^2-(2n\tau)^2}]\,dx\right]u(t-2n\tau) - \frac{(l+\kappa)(1-e^{-at})}{R_0\kappa^2}u(t). \tag{18}$$

This section is closed with the evaluation of $a_{12}(t)$. Since [17]

$$\operatorname{csch} x = 2\sum_{n=1}^{\infty}e^{-(2n-1)x} \qquad \mathrm{Re}\,(x) > 0 \tag{19}$$

(7b) can also be expressed as a series expansion

$$A_{12}(s) = A_{21}(s) = -\frac{2Y_0(0)}{s}\left(1+\frac{l}{\kappa}\right)\sqrt{\frac{(s+\beta)}{(s+a)}} \cdot \sum_{n=1}^{\infty}e^{-(2n-1)\tau\sqrt{(s+a)(s+\beta)}}. \tag{20}$$

Taking the inverse transform of (20) term by term, using once again [15, p. 253, #3.2–64], readily gives

$$a_{12}(t) = a_{21}(t) = -2Y_0(0)\left(1+\frac{l}{\kappa}\right) \cdot \sum_{n=1}^{\infty}\left[e^{-\rho t}I_0[\sigma\sqrt{t^2-(2n-1)^2\tau^2}] + \beta\int_{(2n-1)\tau}^{t} e^{-\rho x}I_0[\sigma\sqrt{x^2-(2n-1)^2\tau^2}]\,dx\right] \cdot u(t-(2n-1)\tau). \tag{21}$$

This terminates the evaluation of the four elements of the step response matrix for a lossy parabolic line. For completeness, we shall derive similar expressions for the lossless parabolic line in the next section.

III. Time-Domain Step Response Matrix of a Lossless Parabolic Line

For the lossless parabolic line, $R_0 = G_0 = 0$, so we have $\alpha = \beta = 0$ from (9) and (10). Consequently, the analytical expressions of the time-domain step response matrix parameters for the lossless case can readily be obtained by first substituting $\alpha = \beta = 0$ in (14), (20) and in a similar series expansion for (7c), and then by evaluating their inverse Laplace transforms term by term using [15, p. 227, #3.0–2] and [15, p. 185, #1–37]. Therefore, the step response matrix becomes

$$a_{11}(t) = 2Y_0(0)\sum_{n=0}^{\infty}\epsilon_n u(t-2n\tau) + \frac{t}{L_0\kappa}u(t) \tag{22a}$$

$$a_{12}(t) = a_{21}(t) = -2Y_0(0)\left(1+\frac{l}{\kappa}\right)\sum_{n=1}^{\infty}u(t-(2n-1)\tau) \tag{22b}$$

$$a_{22}(t) = 2Y_0(0)\left(1+\frac{l}{\kappa}\right)^2\sum_{n=0}^{\infty}\epsilon_n u(t-2n\tau) - \frac{(l+\kappa)}{L_0\kappa^2}tu(t). \tag{22c}$$

IV. Considerations on the Numerical Evaluation

The modified Bessel function $I_0(y)$ was computed using the routine found in [18]. The indefinite integrals in (15), (18), and (21) require numerical techniques for their evaluation and the composite trapezoidal rule (e.g., algorithm *QTrap* in [18]) was found to be well suited for that task.

V. Numerical Results and Discussion

In order to illustrate the possible uses of the matrix obtained in Section II, two examples are presented.

Example 1: We consider the transient response of a single lossy parabolic line driven by a voltage generator with an internal resistance of $R_g = 50\ \Omega$ and terminated in a pure linear resistance $R_L = 100\ \Omega$ (Fig. 1). The line is 0.05 m long and is characterized by the parameters $R_0 = 70\ \Omega$/m, $L_0 = 0.4\ \mu$H/m, $G_0 = 0.0005$ S/m, and $C_0 = 160$ pF/m at $z = 0$. Parameter κ in (2) is chosen so that the high-frequency limit of the "characteristic impedance" $Z_0(s, z)$ is, respectively, 50 Ω at the input port and 100 Ω at the output end of the line. The applied voltage is a unit step with a finite rise time approximated, as in [11], by the incomplete Beta function [18]: $v_g(t) = I_x(4, 4)$. Its rise time, from 10% to 90% of its final value, is very nearly 0.14 ns. Allen's method has been applied to compute the transient response. In order to evaluate the convolution equation (7) in [13], we used a central-difference formula to approximate the derivative of the voltage and an open extended quadrature formula [19].

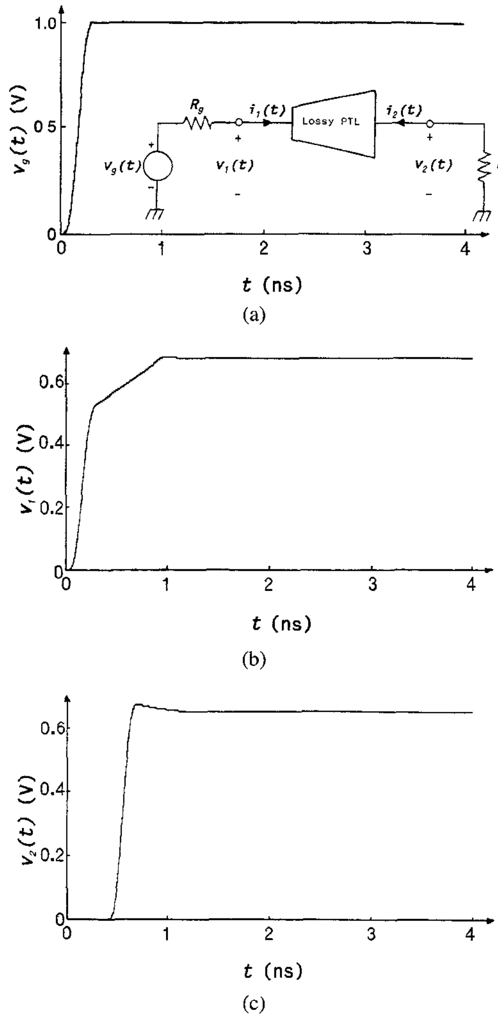
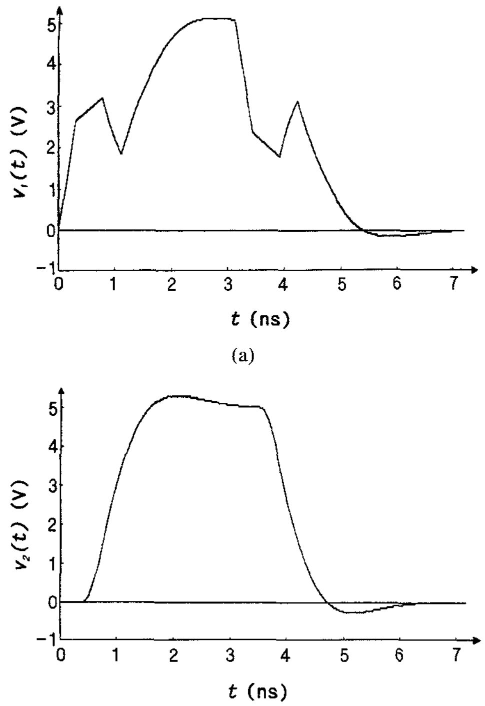


Fig. 1. Step response waveforms for the lossy parabolic line network (Example 1). (a) Generator voltage: $v_g(t) = I_x(4, 4)$ with $x = t/(80\Delta)$, for $0^+ \le t \le 80\Delta$ and $v_g(t) = 1$ for $t > 80\Delta$. $R_g = 50\ \Omega$ and $R_L = 100\ \Omega$. (b) Sending end voltage. (c) Receiving end voltage.

The network of Fig. 1 is composed of a single two-port, the parabolic line, characterized in the time domain by (15), (21), and (18). It is subjected to the boundary conditions $v_1(t) = v_g(t) - R_g i_1(t)$ at $z = 0$ and $v_2(t) = R_L i_2(t)$ at $z = l$. The transient voltages at both ends of the line are shown in Fig. 1(b) and (c) for $0^+ \le t \le 10\tau$, with a time step $\Delta = \tau/100 = 4$ ps [13]. Like the exponential line [11], the parabolic line is not matched for every type of signal by merely forcing the equality between R_g and the high-frequency "characteristic impedance" $Z_0(s, 0) = 50\ \Omega$ at the sending end and, also, between R_L and the high-frequency "characteristic impedance" $Z_0(s, l) = 100\ \Omega$ at the receiving end. Indeed, the effects of multiple reflections can be seen in Fig. 1(b) and (c). As a check, we replaced the lossy line with a *lossless* parabolic line characterized by the step response matrix given in (22): the receiving end voltage $v_2(t)$ for $t = 10\tau = 4$ ns was equal to 0.6665 V, which is quite close to the theoretical value given by the voltage divider equation $R_L/(R_g + R_L) = 0.6667$. In the case illustrated in Fig. 1(c), the voltage $v_2(t)$ for $t = 10\tau = 4$ ns was equal to 0.6448 V, due to the presence of conductor and dielectric losses.



(b)

Fig. 2. Transient response to a trapezoidal input of the circuit in Fig. 1(a) with resistor R_L replaced by an Advanced Schottky TTL inverter (Example 2). (a) Sending end voltage. (b) Receiving end voltage.

Example 2: Next, we consider the same configuration as in Fig. 1, but resistor R_L is replaced by an Advanced Schottky TTL inverter. The voltage generator now delivers a trapezoidal pulse with rise and fall times (between 0% and 100% of their final values) of 0.32 ns, a pulse width of 2.8 ns, and an amplitude of 5 V. We used the simplified model proposed by Shutt-Aine and Mittra [20, Fig. 7(c)] to represent the input of the digital device, i.e., a reverse-biased Shottky diode with a saturation current of 10^{-12} A in parallel with an 8 pF linear capacitor. The discrete circuit model associated with Gear's second-order algorithm [21] was used to compute the voltage and the current across the capacitor at each time step. The operating points of the nonlinear resistor were computed using the iterative piecewise linear method developed by Chua [22], [23]. Fig. 2 shows the voltage response at both ends of the line for $0^+ \leq t \leq 18\tau$, computed using Allen's method, with a time step $\Delta = \tau/100 = 4$ ps [13]. It is interesting to note that even though the line is badly mismatched at the load end because of the logic gate, the voltage $v_2(t)$ has a smoother pulse shape than the sending end voltage $v_1(t)$. Reflections from the load are clearly seen in Fig. 2(a).

VI. Conclusion

The exact analytical expressions of the step response matrix parameters for the lossy parabolic transmission line have been developed in the time domain, therefore extending the range of problems where Allen's method can be applied. This approach can be used to compute the transient response of networks containing lossy parabolic lines subjected to quite general boundary conditions for any time span of interest. For completeness, we also derived the analytical expressions of the step response matrix parameters for the lossless parabolic line. In order to demonstrate the usefulness of the techniques presented in this paper, the response to a physical step and, also, to a trapezoidal input have been presented for a lossy parabolic line terminated in the first case by a linear load and, in the second case, by a nonlinear load.

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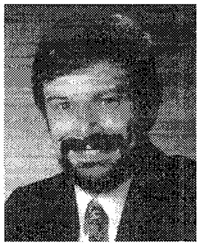

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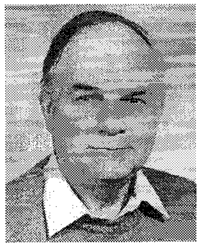

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